Noetherian Rings with Unusual Prime Spectra

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Introduction

Question. *Given a poset X, when can X be realized as the spectrum of a commutative ring R?*

Hochster provided necessary and sufficient conditions for when a poset is the spectrum of a ring [1]. We extend the question by placing restrictions on the ring R, focusing on Noetherian rings. A simple class of spectra to consider are countable ones with uncountable rings.

Question. *Does there exists a nontrivial uncountable Noetherian ring with a countable spectrum?*

This question was unknown until recently when Colbert provided the following result[2]:

Theorem. *There exists, for any $n \geq 0$, an n-dimensional uncountable Noetherian ring with a countable spectrum.*

We extended this result to excellent RLRs.

Algebra Background

Definition. *A **commutative ring**, R, is a set with two 'well-behaved' binary operations, $+$ and $\cdot$, with $0, 1 \in R$ and $a \cdot b = b \cdot a$ for all $a, b \in R$.*

Examples: The integers, $\mathbb{Z}$ and the rationals, $\mathbb{Q}$.

Definition. *An **ideal**, $I \subseteq R$, is additively closed such that for any $a \in I$ and $r \in R$, $ar \in R$. A **prime ideal** is a proper ideal P such that $rs \in P$ implies that $r \in P$ or $s \in P$.*

Examples: $6\mathbb{Z}$ and $n\mathbb{Z}$ are ideals in $\mathbb{Z}$.
$3\mathbb{Z}$ and $p\mathbb{Z}$ are prime ideals in $\mathbb{Z}$.

Definition. *The **spectrum** of a ring, or $\operatorname{Spec} R$, is the set of all its prime ideals.*

Example: $\operatorname{Spec} \mathbb{Z} = \{p\mathbb{Z} : p \in \mathbb{Z}\}$.

Definition. *A **Noetherian** ring is one in which every ideal is finitely generated.*

Definition. *A **local** ring, (R, M) is a Noetherian ring, R, with a unique maximal ideal, M.*

Definition. *Given $P \in \operatorname{Spec} R$, the **localization** of R at P, R_P, inverts all $r \in R \backslash P$.*

Commutative Algebra Background

Definition. *A **regular local ring** (RLR) is a local ring, (R, M), such that M has a minimal set of generators $M = (r_1, \ldots, r_n)$ where $n = \dim R$.*

Definition. *A ring R is **regular** if R_P is a RLR for every $P \in \operatorname{Spec} R$.*

Examples: If k is a field

- k and $k[x_1, \ldots, x_n]$ are regular rings
- k and $k[[x_1, \ldots, x_n]]$ are RLRs

Definition. *A local ring (R, M) is **excellent** if*

1. *For all $P \in \operatorname{Spec} R$, $\widehat{R} \otimes_R L$ is regular for every finite field extension L of R_P/PR_P.*
2. *R is universally catenary*

A sufficient condition for excellent in this context:

Lemma. *Given A with $\widehat{A} = T = \mathbb{Q}[[x_1, \ldots, x_n]]$, A is excellent if for each $P \in \operatorname{Spec} A$ and $Q \in \operatorname{Spec} T$ with $Q \cap A = P$, $(T/PT)_Q$ is a RLR.*

Theorem

Theorem 1 (AM). *There exists, for any $n \geq 0$, an n-dimensional uncountable excellent regular local ring with a countable spectrum.[3]*

We prove existence constructively, creating a ring, B, such that

$$\mathbb{Q}[x_1, \ldots, x_n] \subset B \subset \mathbb{Q}[[x_1, \ldots, x_n]].$$

Since $\mathbb{Q}[x_1, \ldots, x_n]$ has a countable spectrum, we use this to bound the cardinality of the spectrum of our ring, meanwhile adjoining uncountably many elements from $\mathbb{Q}[[x_1, \ldots, x_n]] = T$.

Outline

Outline of Construction

1. The Base Ring, S
   - $\mathbb{Q}[x_1, \ldots, x_n] \subset S \subset \mathbb{Q}[[x_1, \ldots, x_n]]$.
   - If $s \in pT$, then $pu \in S$ for some $u \in T$.
   - S is excellent, countable, and $\widehat{S} = T$
2. Uncountability
   - Adjoin uncountably many units $u \in T$
   - Preserve the spectrum's cardinality
3. Excellence
   - Adjoin elements so that $bT \cap B = bB$.
   - Prove B is excellent.

The Construction

STEP 1: The Base Ring

Starting with $R_0 = \mathbb{Q}[x_1, \ldots, x_n]$, we adjoin elements t_i from $\mathbb{Q}[[x_1, \ldots, x_n]] = T$ to obtain the desired properties for our base ring.

$$R_0 \subseteq R_0[\{t_i\}] \subseteq R_1 \subseteq R_1[\{t_j\}] \subseteq R_2 \subseteq \cdots$$

Define

$$S = \bigcup_{n=0}^{\infty} R_n.$$

Lemma. *The ring S is excellent, has completion T, is countable and for any $s \in S \cap pT$ for a prime $p \in T$, there exists $u \in T$ such that $pu \in S$.*

STEP 2: Uncountability

To S we adjoin uncountably many units from T,

$$S = S_0 \subset S_0[u_0] = S_1 \subset S_1[u_1] = S_2 \subset \cdots$$

such that we maintain the following property:

Property $(S^\star)$. *A ring $R \supseteq S$ has Property $(S^\star)$ if whenever $P \in \operatorname{Spec} T$ and $P \cap S = (0)$, $P \cap R = (0)$.*

Define

$$A = \bigcup_{i=0}^{\infty} S_i.$$

Choosing uncountably many u_i, A is uncountable but, by Property $(S^\star)$, has a countable spectrum.

The Construction (cont.)

STEP 3: Excellence

To A we adjoin elements from T,

$$A = A_0 \subseteq A_0[\{t_i\}] = A_1 \subseteq A_1[\{t_i\}] = A_2 \subset \cdots$$

and define

$$B = \bigcup_{n=0}^{\infty} A_n,$$

choosing the t_i's such that

Lemma. *The ring B satisfies*

- *$bT \cap B = bB$ for any $b \in B$, and*
- *B has Property $(S^\star)$*

Using factoring property of S we have

Lemma. *All ideals of B are extended from S.*

From this we can show that all finitely generated ideals are closed up and thus:

Lemma. *The ring B is Noetherian, has completion $\widehat{B} = T$, and B is a RLR.*

Finally, using that S is excellent we have

Lemma. *The ring B is excellent.*

Since $\widehat{B} = T$, and $\dim T = n$, B is n-dimensional. Thus, combining the above lemmas we have:

Theorem. *There exists, for any $n \geq 0$, an n-dimensional uncountable excellent regular local ring with a countable spectrum.*

References

[1] M. Hochster. Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, 142:43–60, 1969.

[2] Cory Colbert. Enlarging localized polynomial rings while preserving their prime ideal structure. *J. Algebra, to appear*.

[3] S. Loepp and A. Michaelsen. Uncountable n-dimensional excellent regular local rings with countable spectra. *Under review, on arXiv.*