Noetherian rings with unusual prime ideal structures

Anya Michaelsen

Williams College

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Background

Remark

In this talk, a ring is a commutative ring with unity.

Definition

An **ideal** is an additively closed subset I of a ring R, such that for $a \in I$, $r \in R$, $ra \in I$. A **prime ideal** is a proper ideal P such that if $rs \in P$, then either $r \in P$ or $s \in P$.

Definition

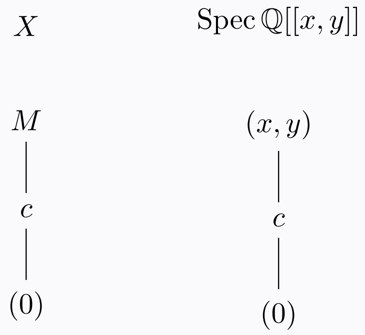
Given a ring R, the **spectrum** of a R, denoted $\operatorname{Spec} R$, is the set of all its prime ideals.

Previous Results

Question

Given a poset X, when can X be realized as the spectrum of a (commutative) ring R?

Example



Previous Results

Theorem (Hochster)

Provided necessary and sufficient conditions for when a poset is the spectrum of a ring.

Question

Given a poset X, when can X be realized as the spectrum of a (commutative) ring R with [property]?

Definition

A **Noetherian** ring is one in which every ideal is finitely generated.

Previous Results

Question

Does there exist a (nontrivial) uncountable Noetherian ring with a countable spectrum?

Ring	Uncountable?	Countable Spec?
$\mathbb{Q}[x,y]$	no	yes
$\mathbb{Q}[[x,y]]$	yes	no

Theorem (Colbert, 2016)

There exists an n-dimensional uncountable Noetherian ring with countable spectrum for any $n \geq 0$.

Regular local rings

Definition

A **regular local ring (RLR)** is a local ring, (R, M), such that M has a minimal set of generators $M = (r_1, \ldots, r_n)$ where $n = \dim R$.

Definition

A ring R is **regular** if R_P is a RLR for every $P \in \operatorname{Spec} R$.

Examples: If k is a field

- k and $k[x_1, \ldots, x_n]$ are regular rings
- k and $k[[x_1, \ldots, x_n]]$ are RLRs

Background

Definition

A Noetherian local ring $(A, A \cap M)$ is **excellent** if

1. For all $P \in \operatorname{Spec} A$, $\widehat{A} \otimes_A L$ is regular for every finite field extension L of A_P/PA_P.
2. A is universally catenary

Lemma

Given A *with completion* $T = \mathbb{Q}[[x_1, \ldots, x_n]]$, A *is excellent if for each* $P \in \operatorname{Spec} A$ *and* $Q \in \operatorname{Spec} T$ *with* $Q \cap A = P$, $(T/PT)_Q$ *is a regular local ring* (RLR).

Result

Theorem (AM)

There exists an n-dimensional uncountable excellent regular local ring with a countable spectrum for any $n \geq 0$.

Construction

Given $n \geq 2$,

$$\mathbb{Q}[x_1, \ldots, x_n] \subset B \subset \mathbb{Q}[[x_1, \ldots, x_n]] = T$$

$$\operatorname{Spec} \mathbb{Q}[x_1, \ldots, x_n]:$$

$$(x_1, \ldots, x_n)$$

$$|$$

$$\aleph_0$$

$$|$$

$$\vdots$$

$$|$$

$$\aleph_0$$

$$|$$

$$(0)$$

Construction

Theorem (AM)

There exists an n-dimensional uncountable excellent regular local ring with a countable spectrum for any $n \geq 0$.

1. The Base Ring, S
   - $\mathbb{Q}[x_1, \ldots, x_n] \subset S \subset \mathbb{Q}[[x_1, \ldots, x_n]] = T$.
   - If $s \in pT \in \operatorname{Spec} T$, then $pu \in S$ for some $u \in T$.
   - S will be excellent, countable, with $\widehat{S} = T$
2. Uncountability
   - To S we will adjoin uncountably many units from T
   - Preserve the cardinality of the spectrum

Construction

Theorem (AM)

There exists an n-dimensional uncountable excellent regular local ring with a countable spectrum for any $n \geq 0$.

3. Excellence
   - Adjoin elements so that for $b \in B$, $bT \cap B = bB$.

Lemma

Every finitely generated ideal of B is extended from S. Hence, $IT \cap B = IB$ for finitely generated ideals.

Lemma

The ring B is Noetherian with completion T. Hence B is a RLR.

Construction

3. Excellence
   - Adjoin elements so that for $b \in B$, $bT \cap B = bB$.

Lemmas

- Every finitely generated ideal of B is extended from S.
- $IT \cap B = IB$ for finitely generated ideals $I \subseteq B$.
- The ring B is Noetherian with completion T. Hence B is a RLR and has dimension n.

Theorem (AM)

There exists an n-dimensional uncountable excellent regular local ring with a countable spectrum for any $n \geq 0$.

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