UC Berkeley Qualifying Exam

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Complex Analysis Study Guide

Minor topic: Complex Analysis (Analysis)

References: Stein and Shakarchi, *Complex Analysis*, Chapters 1-3

- **Complex functions:** holomorphic, meromorphic, Cauchy-Riemann equations, Liouville's theorem, Taylor and Laurent series
- **Complex integration:** Cauchy's theorem, Cauchy's integral formula, residue theorem, argument principle, Rouché's theorem, Morera's Theorem, maximum modulus principle
- **Fundamental Theorem of Algebra:** Statement and proof.

Contents

Memorization (– key terms –)

Chapter 1: Preliminaries on Complex Analysis

1.1 Complex Numbers and Complex Plane

1 region, connected open/closed set, path connected

region = open and connected set in $\mathbb{C}$.

connected open/closed cannot be written as a union of proper open/closed subsets.

path connect = all points can be connected by a path

2 compact, diameter of a set

compact = closed and bounded $\iff$ every sequence has subsequence conv. to a point in Ω $\iff$ open coverings have finite subcover

$\text{diam}(\Omega) = \sup_{z,w\in\Omega} |z - w|$

3 complex convergence of a series and Cauchy sequences

$z_n \to w$ iff $\lim |z_n - w| \to 0$. converges $\iff$ real and complex parts converge

Cauchy Seq- $|z_n - z_m| \to 0$ and $n, m \to \infty$

4 absolutely and uniform convergence

absolute convergence $\sum |z_n|$ converges (in the real sense)

convergence of functions f_n $\forall x$ and $\epsilon > 0$ there is an N s.t. for $n \geq N$ $|f_n(x) - f(x)| < \epsilon$ then $f_n \to f$.

uniform convergence for f_n functions for every ϵ there is an N s.t. for $n \geq N$ $|f_n(x) - f(x)| < \epsilon$ $\forall x$.

If $f_n \to f$ uniformly, then $\lim_n \int_a^b f_n dx = \int_a^b f dx$.

1.2 Complex Functions on $\mathbb{C}$

5 holomorphic function

the limit $\lim_{h\to\infty} \frac{f(z+h)-f(z)}{h}$ exists for all possible ways $h \to 0$.

6 Cauchy-Riemann Equations

If $f(x, y) = u(x, y) + iv(x, y)$ then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

7 power series for $e^z, \sin(z), \cos(z)$

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\quad$ $\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ $\quad$ $\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$

8 radius of convergence

$|z| < R$ the series converges absolutely, $\quad$ $|z| > R$ the series diverges, $\quad$ $|z| = R$ is undetermined

9 Hadamard formula

$1/R = \limsup |a_n|^{1/n}$

10 analytic function

a function that has a power series expansion at a point in a neighborhood of it ($\iff$ holomorphic)

1.3 Integration along curves

11 smooth/piece-wise smooth curve, length of a curve

$z(t) : [a, b] \to \mathbb{C}$ parameterized curve. Smooth means $z'(t)$ exists and continuous, piece-wise if can be chopped up to be smooth on pieces.

length$(\gamma) = \int_a^b |z'(t)|dt$.

12 closed curve, simple curves

closed - endpoints match (i.e. is a loop)

simple - non-intersecting

13 path integral $\int_\gamma f(z)dz$

let $z(t)$ be a parameterization on $[a, b]$ then $\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$.

14 primitive of a function

a function $F(z)$ that is holomorphic and $F'(z) = f(z)$ in the specified region.

Chapter 2: Cauchy's Theorem and Its Applications

2.1 Goursat's Theorem

15 Goursat's Theorem

If f is holomorphic on an open set containing a triangle T and its interior, then

$$\int_T f(z)dz = 0$$

2.2 Local Existence of Primitives & Cauchy Theorem in a Disc

16 Cauchy's theorem on a disc

If f is holomorphic on a disc and its boundary, then

$$\int_D f(z)dz = 0$$

17 toy contours

regions in $\mathbb{C}$ with clearly defined 'interiors' like a circle, triangle, square/rectangle, keyhole

2.3 Evaluation of some Integrals

18 keyhole contour (and its limit)

corridor gets narrow and cutout circle shrinks around the point

19 semi-cirlce contour without origin

outer radius grows to infinity and inner radius shrinks to the origin point

2.4 Cauchy's Integral Formulas

20 Cauchy's integral formula for f

f holomorphic on an open set containing disc D and its closure
C be the boundary circle (with positive orientation)

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

for any $z \in D$ (integral vanished outside of D)

21 proof sketch for Cauchy's integral formula for f

key hole contour with z removed
limit of the corridor gets two circles by cancellation $\int_C + \int_{C_\varepsilon} = 0$
rewriting to get $f(z)$ in the numerator, can explicitly compute inner circle
solve for $\int_C$ to get desired result.

22 Cauchy's integral formula for $f^{(n)}$

f holomorphic on an open set containing circle C and its interior

$$f(z)^{(n)} = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

for any z in the inerior of C.
Key idea: f being holomprhic $\implies$ infinitely differentiable

23 Cauchy inequalities

f holomprhic in an open set containing closure of disc D centered at z_0 with radius R

$$|f^{(n)}(z_0)| \leq \frac{n!||f||_C}{R^n}$$

where $||f||_C = \sup_{z\in C} |f(z)|$ (suprememum on the boundary of the disc)
proof by cauchy integral formula for derivative and taking magnitude

24 power series representation of holomorphic functions

If f holomorphic on a disc (and its closure) around a point z_0, then f has power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

for all $z \in D$ the disc around the point.
Proof Idea: take the cauchy integral formula for $f(z)$ and expand $1/\zeta - z$ in terms of z_0 then by uniform convergence interchange the sum with the integral.

25 Liouville's Theorem

If f is entire (holomorphic on $\mathbb{C}$) and bounded ($|f| \leq M$) then f is constant!!

26 Fundamental Theorem of Algebra and its proof

Let $P(z)$ be a nonconstant polynomial in $\mathbb{C}[z]$. Suppose it has no roots in $\mathbb{C}$, then $1/P(z)$ has no poles and so is entire. To show it is bounded we will split $\mathbb{C}$ into two regions. To show $1/P(z)$ bounded, meaning $|1/P(z)| \leq M$ for some M, we must show that $|P(z)| \geq 1/M$ so is bounded below.

Dividing $P(z)/z^n = a_n + \frac{1}{z}(Q(1/z))$. As $z \to \infty$, the second term goes to 0, so for some R, it is less than $|a_n|/2$ and so $|P(z)/z^n| \geq |a_n|/2$ so where $|z| > R$, $|P(z)| \geq |z|^n|a_n|/2$. So in this region, $|P(z)| \geq R^n|a_n|/2$ so is bounded from below outside $|z| > R$. Inside, it is a finite region with no poles and it continuous, so will be bounded below also. (if not, there will be a sequence of points of f with $f(z_n) \to 0$ which means that $f(\lim z_n) = 0$).

since $1/P(z)$ is entire and bounded, it is constant and so $P(z)$ is constant, a contradiction. So we must have some roots.

2.5.1 Morera's Theorem

27 Morera's Theorem

Morera's Theorem. *Let f be a continuous function in the open disc D. If for every triangle T contained in D,*

$$\int_T f(z)dz = 0$$

then f is holomorphic.

Chapter 3: Meromorphic Functions and the Logarithm

3.1 Zeros and Poles

28 zeros, order/multiplicity

$f(z)$ has a zero at z_0 when $f(z_0) = 0$

order of z_0 is n where $f(z) = (z - z_0)^n h(z)$ near z_0, where $h(z)$ is holomorphic and $h(z_0) \neq 0$.

29 poles, order/multiplicity

f has a pole at z_0 when $1/f$ extended to 0 at z_0 is holomorphic.

multiplicity is the zero multiplicity of $1/f$ at z_0, that is $f(z) = (z - z_0)^{-n}h(z)$, $h(z)$ is holomorphic.

30 simple poles/zeros

simple zeros/poles have order 1

31 principal part of f

Given f with a pole of order n at z_0, we have the expansion near z_0

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + G(z)$$

where $G(z)$ is holomorphic.

The principal part is where f has the pole, i.e. $f(z) - G(z)$.

32 residue of f at pole z_0

Given f with a pole of order n at z_0, we have the expansion near z_0

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + G(z)$$

where $G(z)$ is holomorphic.

The residue is a_{-1}, i.e. the coefficient for $(z - z_0)^{-1}$ in the expansion of f.

33 limit formula for (simple) poles

simple pole: $\operatorname{res}_{z_0} f = \lim_{z \to z_0}(z - z_0)f(z)$

general pole: $\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z - z_0)^n f(z)$

3.2 The Residue Formula

34 residue formula

f holomorphic except for poles at $z_0, \ldots, z_n$ in a region Ω containing C,

$$\boxed{\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} \operatorname{res}_{z_i} f}$$

3.3 Singularities and Meromorphic Functions

35 isolated singularities

Removable if we can define $f(z_0)$ is such a way to make f holomorphic including z_0.

Pole if $1/f(z)$ has removable singularity with $1/f(z_0) = 0$ making $1/f$ holomorphic near z_0

Essential Singularity not a pole or removable singularity

36 meromorphic functions

Meromorphic functions: functions have poles at $z_0, z_1, z_2, \ldots$ points and holomorphic elsewhere with no limit points of z_i's in the region.

37 example of essential singularity

$f(z) = e^{1/z}$ has essential singularity at $z = 0$.

38 Laurent Series

A laurent series about z_0 is $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ for some region $\{r < |z - z_0| < R\}$.

3.4 The Argument Principles and Applications

39 Argument Principle

f is meromorphic in Ω containing a cirlce C and it's interior with no poles/zeros along C then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \#\text{ of zeros of } f \text{ inside } C - \#\text{ of poles of } f \text{ inside } C$$

both zeros and poles counted *with multiplicity*

40 Rouche's Theorem

Let f and g be holomorphic in Ω containing a circle C and $|f(z)| > |g(z)|$ for all $z \in C$. Then f and $f + g$ have the same number of zeros inside C.

41 Open Mapping Theorem

If f is holomorphic and nonconstant on some region Ω then f is an open map.

42 Maximum Modulus Principle

If f is holomorphic and nonconstant on some region Ω then f does not attain a max modulus on Ω.

Complex Analysis Quals Questions (– best questions –)

Chapter 1: Preliminaries on Complex Analysis

1.1 Complex Numbers and the Complex Plane

1 Show there is no total ordering on $\mathbb{C}$.

Total ordering - $a < b$ or $a > b$ or $a = b$ for all a, b. $a < b \implies a + c < b + c$ for all a, b, c if $a < b$ and $0 < c$ then $ac < bc$.

Case 1: $i < 0$ $\quad 0 < -i$ so $0 < (-i)^2 = -1$ meaning $1 < 0$ so then $-i < 0$ so $0 < i$ so $0 < -i^2 = 1$.

Case 2: $i > 0$ $\quad i^2 = -1 > 0$ then $0 > 1$ but also $-i > 0$ so then $-i^2 = 1 > 0$

2 If Ω is open, show that Ω is path connected if and only if it is connected.

Assume open and pathwise-connected: If disconnected, $\Omega = \Omega_1 \sqcup \Omega_2$. Take $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ which are path connected by γ.

Parameterize by $z : [0, 1] \to \Omega$ and take $t^* = \sup_t \{t : z(s) \in \Omega_1 : 0 \leq s < t\}$.

contradiction with $z(t^*)$: first, $z(t^*) \in \Omega_1$. Well Ω_1 is open, so there is a region around $z(t^*)$ contained in Ω_1. The path contains a nearby point for $t^* < t^{**}$ in this region, but then t^* is not the supremum unless $t^* = 1$ but $z(1) = \omega_2 \notin \Omega_1$.

Assume open and connected: Take $\omega_1 \in \Omega$. Define $\Omega_1 \subset \Omega$ as the set of all point path-connected to ω

Show Ω_1 is open, closed, and nonempty. (all points in a disc are path connected!)

Open: take any point, since Ω is open, there is a region around this point in Ω, and within an open disc all points are connected, so these points are also in Ω_1, hence it is open.

Closed: Take a point not in Ω_1, since Ω is open there is a small disc containing the pt. If any of those points are in Ω_1, then extend path to the point, contradiction so then the disc is in Ω_1^C which is open.

Nonempty: well $\omega_1 \in \Omega_1$.

Then $\Omega = \Omega_1 \sqcup \Omega_2$ and Ω is connected so $\Omega = \Omega_1$, showing that Ω is path connected.

3 Example of a set that is connected but not path connected. Any examples of path-connected but not connected?

Connected but *not* path-connected:

Topologist's Sine Curve: all points $(x, \sin(1/x))$ with the origin.

Connected: points get arbitrarilty close to origin so cannot separate it (all other points lie on a continuous function and so cannot be separated)

Not-path-connected: cannot find a path between origin and any point on the sin curve. Suppose there is a path γ given by $f : [0, 1] \to \mathbb{C}$ then by $\varepsilon - \delta$ there is a region close to 0 such that $|f(x) - (0, 0)| < 1/2$ but can always come a little closer on the sine curve to get back to $(x, 1)$ which will exceed the 1/2 bound.

Path-connected $\implies$ connected: If disconnected, take a path γ from $\omega_1 \in \Omega_1$ to $\omega_2 \in \Omega_2$. Then $\gamma : [0, 1] \to \Omega$ is continuous, and so $\gamma^{-1}(\Omega_1)$ and $\gamma^{-1}(\Omega_2)$ for a disjoint cover of $[0, 1]$, which is connected, contradiction.

1.2 Functions on the Complex Plane

4 Show that a continuous function on a compact set is bounded and attains a max/min.

f cont. on Ω compact (closed and bounded). Take $x \mapsto |f(x)|$ which is continuous $\Omega \to \mathbb{R}$.

First, show that this is bounded. If not, then there is a sequence $f(x_n)$ that is unbounded, but Ω is compact, so there is a convergent subsequence converging to a point $y \in \Omega$. Thus there is a convergence subsequence $f(x_k) \to f(y)$ which is bounded, contradiction.

Second, this converges to a point $f(x)$ for some x. Taking the least upper bound, take a sequence of points getting closer to the bound $f(x_n)$. Again, a subsequence converges to z and $f(x_n) \to f(z)$ which must also converge to the least upper bound, i.e. a maximum of f.

5 Give an example of a function and a set on which the function does not attain a max/min.

$f(z) = z$ on $\mathbb{C}$.

6 Show that holomorphic functions are continuous.

Holomorphic - has complex derivative , Continuous - ε-δ definition with complex distances

Given that f is holomorphic, the limit of $\lim_{h\to 0} \frac{f(z_0+h)-f(z)}{h}$ exists, so we can write $f(z_0+h) - f(z) = Lh + h\Psi(h)$ where $\Psi \to 0$ as $h \to 0$. Then for $\varepsilon > 0$ choose δ such that $|Lh + h\Psi(h)| < \varepsilon$ for all $|h| < \delta$. Then $|z - z_0| < \delta$ means $z = z_0 + h$ for $|h| < \delta$ gives the desired ε claim.

7 Derive the Cauchy-Riemann Equations.

Take $f(x, y) = u(x, y) + iv(x, y)$ and derivative twice, once by $(x + h, y) \to (x, y)$ (real approach) and once by $(x, y + h) \to (x, y)$ (complex approach).

Express derivative both times in terms of partial derivatives of u and v (keeping track of real/maginary) then equate real and imaginary parts!

8 Give an example of a non-holomorphic function that is differentiable as $f(x, y)$.

$f(z) = \overline{(z)}$ ie $f(x, y) = x - iy$ so $u = x$ and $v = -y$. These are diff but do not satisfy Cauchy-Riemann. Limit does not exist when approaching by $h = h \in \mathbb{R}$ versus $h = ih$, one gives $+1$ and one gives -1.

9 Find the radius of convergence of e^z, $\sin z$, $\sum_n z^n$, $\cos z$.

radius of convergence = R such that for all $|z| < R$ the function/series converges.

$e^z = \sum_{n\geq 0} \frac{z^n}{n!}$, radius of convergence $R = \infty$.

show absolute convergence for $\sum_{n\geq 0} |z^n/n!| = \sum_{n\geq 0} |z|^n/n!$.

ratio test... $\lim \frac{|z|^{n+1} n!}{|z|^n (n+1)n!} = \lim \frac{|z|}{n!} = 0 < 1$ so this converges for all $|z|$ and thus all $z \in \mathbb{C}$.

$\sin(z) = \sum_{n\geq 0} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ radius of convergence $R = \infty$ (similar for cos)

show absolute convergence ... ratio test $\lim \frac{|z|^{2n+3}(2n+1)!}{|z|^{2n+1}(2n+3)(2n+2)(2n+1)!} = \lim \frac{|z|^2}{(2n+3)(2n+2)} = 0 < 1$ so this converges for all $|z|$ and thus all $z \in \mathbb{C}$.

$\sum_{n\geq 0} z^n$ radius of convergence $R = 1$

ratio test for absolute convergence $\lim |z|^{n+1}/|z|^n = \lim |z| = |z| < 1$ converges absolutely.

10 State and prove Hadamards formula for the radius of convergence.

Formula: Let R be the radius of convergence of a series $\sum_{n\geq 0} a_n z^n$, then

$$1/R = \limsup |a_n|^{1/n}$$

where $1/0 = \infty$ and $1/\infty = 0$.

Proof:

If $|z| < R$ then converges.... $L = 1/R$ so that $|z|L < 1$ choose ε such that $(L+\varepsilon)|z| = r < 1$. By defn $L = \limsup |a_n|^{1/n}$ so $|a_n|^{1/n} \leq L+\varepsilon$. Then $|a_n||z|^n \leq (L+\varepsilon)^n |z|^n = r^n$ and $r < 1$. So the series $\sum r^n$ converges so too does $\sum a_n z^n$.

If $|z| > R$ then diverges... $|z|L > 1$ so choose $\varepsilon > 0$ with $(L - \varepsilon|z|) = r > 1$. Then by lim sup there exists a sequence of n_k such that $|a_{n_k}|^{1/n_k} > L - \varepsilon$. So take this subseries which is greater than $\sum r^n$ which diverges to infinity.

11 Show that the derivative of $f(z) = \sum a_n z^n$, $f'(z) = \sum n a_n z^n$, has the same radius of convergence.

Use Hadamard formula. $\lim n^{1/n} = 1$ and given that the derivative is term by term differentiated with coefficients na_n this will have the same lim sup and thus the same radius of convergence.

12 Explain why complex differentiable is stronger than real differentiable.

Holomorphic means that a function is differentiable, but all holomorphic functions are inifnitely differentiable. For real functions, there are differentiable functions that are not infinitely differentiable, such as, $x \cdot |x|$ (derivative is $2|x|$). So we see that complex differentiation is much stronger.

Complex differentiation also implies the function is analytic (has a power series expansion at every point), which is not true for real functions.

One reason for this is in the definition of complex differentiation, we account for infinitely many ways that $h \to 0$ wheras for $h \in \mathbb{R}$ it can approach 0 only from above or below.

1.3 Integration along curves

13 Give a parameterization of the unit circle and integrate $f(z) = 1/z$ along this.

Take γ represented by $z(t) = e^{it}$ for $0 \leq t \leq 2\pi$ postiively orientied closed loop of the unit circle.

$$\int_\gamma f(z)dz = \int_0^{2\pi} f(z(t))z'(t)dt = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}dt = \int_0^{2\pi} idt = 2\pi i$$

14 Show that $f(z) = 1/z$ does not have a primitive defined on $\mathbb{C}$.

primitive = holomorphic function F such that $F' = f$.

If f is continuous (which it is) and has a primitive (which it doesnt), then taking γ along the unit circle, then $\int_\gamma f(z)dz = 0$ however by direct computation (question above) this is $2\pi i \neq 0$ so f cannot have a primitive.

15 Compute the integral of $f = z^n$ for any n along the curve $\gamma = e^{it}$ $0 \leq t \leq 2\pi$.

$\int_\gamma f(z)dz = \int_\gamma z^n dz = \int_0^{2\pi} e^{itn} ie^{it}dt = i\int_0^{2\pi} e^{it(n+1)}dt = i\left[\frac{1}{i(n+1)} e^{it(n+1)}\Big|_0^{2\pi}\right] = \frac{1}{n+1}(e^{2\pi i(n+1)} - e^0) = 0$

Chapter 2 : Cauchy's Theorem and Its Applications

2.1 Goursat's Theorem

16 State and sketch a proof of Goursat's Theorem.

Goursat's Theorem: If f is holomorphic on an open set containing a triangle T and its interior, then

$$\int_T f(z)dz = 0.$$

Break T into four triangles by bisecting each side. Choose the triangle that maximizes $|\int_{T'} f(z)dz|$ so that $|\int_T f(z)dz| \leq 4|\int_{T_n} f(z)dz|$.

want to use the diameter of T_n and perimeter to bound the size here.

Basic Idea: f differs from a function with a primitive (contstant and linear pieces) by a function that goes to zero on small regions. Taking smaller and smaller triangles we leverage this asymptotic behavior.

Since f is holomorphic, $f(z) = f(z_0) + f'(z_0)(z - z_0) + \phi(z)(z - z_0)$ where $\phi \to 0$ and $z \to z_0$. Using the primitives for $f(z_0)$ (a constant) and $f'(z_0)(z - z_0)$ a linear polynomial, we have $\int f(z)dz = \int \phi(z)(z - z_0)$. The max value of this is $\epsilon_n d_n$ where ϵ_n is max of ϕ on T_n and $d_n = diam T_n = d_0/2^n$.

$$|\int_{T_n} f(z)dz| = |\int_{T_n} \phi(z)(z - z_0)dz| \leq \|\phi(z)\|_{T_n} \|z - z_0\|_{T_n} p_n = \epsilon_n d_n p_n = 4^{-n} \epsilon_n d_0 p_0$$

so then $|\int_T f(z)dz| \leq \epsilon_n d_0 p_0 \to 0$ as $\epsilon_n \to 0$ as $n \to \infty$ so the integral is 0.

2.2 Local primitives and Cauchy on a disc

17 Give a sketch of why holomorphic functions have primitives in discs. Where does Goursat's Theorem come in?

Take the path γ from origin $(0,0) \to (x,0) \to (x,y)$ [representing $z = x + iy$]

define $F(z) = \int_\gamma f(z)dz$, this will be the primitive.

Claims: holomorphic on the disc and $F'(z) = f(z)$

$z + h$ and z paths can be split into the direct path and triangles/rectangles, which are 0 by Goursat's.

Then as $h \to 0$ the path length goes to 0 we can control the extra part showing derivative is $f(z)$

18 What is a toy contour? Give some examples. Why do we use them?

Toy contours are regions in $\mathbb{C}$ where 'interior' is easily defined.

Examples: circles, triangles, squares and rectangles, keyholes and cutouts of regular shapes

We use these to develop Cauchy results in simpler (more common) cases and make intuitive sense of these results

Favorite toy contour: keyhole, its so clever and funny! (also looks like Gallifreyan writing)

19 How do we define the primitive for Cauchy's theorem in toy contours (not just discs)?

Use right angled path to connect any point to a fixed base point, any two paths will differ by rectangles/triangles, which in an integral will vanish by Goursat's so this is well-defined.

20 State Cauchy's Theorem. How would you prove it using Green's Theorem?

Cauchy's Theorem: *If f holomorphic on a region Ω and γ a smooth closed loop in Ω then* $\int_\gamma f(z)dz = 0$.

Proof by Green's Theorem:

Write $f(z) = u + iv$ and $dz = dx + idy$ then

$$\int_\gamma f(z)dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy)$$

(Can also make more rigorous than "$dz = dx + idy$" by taking a parametrization of γ and then multiplying everything out and recognizing $\int u(a(t), b(t))a'(t)dt = \int_\gamma udx$)

Green's Theorem: $\int_C (Ldx + Mdy) = \iint_D ((\frac{\partial M}{dx} - \frac{\partial L}{dy})dxdy$

So using Cauchy-Riemann Equations:

$\int_\gamma (udx - vdy) = \iint_D (-\frac{\partial v}{dx} - \frac{\partial u}{dy})dxdy = -\iint_D (\frac{\partial v}{dx} + \frac{\partial u}{dy})dxdy = -\iint_D (0dxdy = 0$

$\int_\gamma (vdx + udy) = \iint_D ((\frac{\partial u}{dx} - \frac{\partial v}{dy})dxdy = \iint_D ((\frac{\partial u}{dx} - \frac{\partial v}{dy})dxdy = \iint_D (0dxdy = 0$

So $\int_\gamma f(z)dz = 0$.

21 What is Goursat's Theorem? How can you use Goursat's Theorem to prove Cauchy's Theorem without assuming f is *continuously* differentiable (only diff. at every point)?

Goursat's Theorem: If f differentiable everywhere, then $\int_T f(z)dz = 0$ for any triangle T in the region that f is differentiable.

Goursat's $\implies$ Cauchy's Theorem:

Idea: Use Goursat to construct a primitive for f in the region, using the primitive $\int_\gamma f dz = 0$.

Primitive is $F(z) = \int_{\gamma_z} f(\omega)d\omega$ where γ_z is a taxicab curve from a fixed point z_0 to z. Using Goursat's, show that this is differentiable and it's derivative is f (by canceling the curves to z and $z+h$ using triangles/rectangles) and so F is the primitive.

2.3 Evaluation of some Integrals

22 How would you compute $\int_0^\infty \frac{1-\cos(x)}{x^2}dx$?

1) rewrite as the real part of some complex function, in this case

$$f(z) = \frac{1-e^{iz}}{z^2} \qquad \left(\text{Re}(f) = \frac{1-\cos(x)}{x^2}\right)$$

2) singularity at $z = 0$ so want to avoid the origin

contour: semi-circle with origin removed

3) this is even function, so could take $\int_{-\infty}^{\infty}$ and divide by 2

evaluate the pieces of the integral, and take limits so $R \to \infty$ and $\varepsilon \to 0$

2.4 Cauchy's Integral Formulas

23 State Cauchy's Integral Formula. How is it derived?

Cauchy's Integral Formula: *For f holomorphic on a disc D and its closure (with boundary C)*

$$f(x) = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{\omega - z} d\omega$$

for any $z \in D$.

Proof Sketch:

keyhole contour along C with z cutout.

let corridor go to 0 to get two circles in the integral

explicity compute the inner circle by rewriting $f(\omega)$ and then solve for bigger integral.

24 What are some consequences of Cauchy's Integral Formula(s)?

- bounds on derivative magnitude (so called Cauchy's Inequalities)

  take Cauchy's integral formulas for derivatives and bound the size

- holomorphic $\implies$ analytic (power series)

  take the cauchy integral formula for $f(z)$ and expand $1/\zeta - z$ in terms of z_0

  by uniform convergence interchange the sum with the integral.

- Liouville's Theorem: entire + bounded $\implies$ constant

  show $f' = 0$ everywhere via cauchy inequalities

  $|f'(z)| \leq B/R$ for global bound B and any radius R, let $R \to \infty$ gives $f' \to 0$

- Fundamental Theorem of Algebra

  prove by Liouville's Theorem

- uniqueness of analytic continuations

  power series expansion in a small region yields information about the global functions

25 State and prove Liouville's Theorem. What if f is entire and $f^{(n)}$ is bounded? What happens if only $\text{Im}(f)$ is bounded?

Louiville's Theorem: *If f is entire and bounded, f is contant.*

Proof:

f bounded, means that $|f| \leq B$ for some B on all of $\mathbb{C}$, hence

By Cauchy Integral formula,

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega - z)^2} d\omega\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^2} Rie^{i\theta} d\theta\right| \leq \frac{1}{2\pi}|\frac{f(z)}{R}||_C|2\pi = \frac{||f||_C}{R} \leq \frac{B}{R}$$

for any R, letting $R \to \infty$, we have that $|f'| \to 0$ for every point $z_0 \in \mathbb{C}$.

$f' = 0$ implies that f is constant

(because $0 = \int_\gamma f' dx = f(\gamma_1) - f(\gamma_0)$ so all values of f are the same, i.e. constant)

Modification 1: f entire, $f^{(n)}$ bounded

Then Apply Louiville to $f^{(n)}$ (which is also entire since f is inifnitely diff.) so $f^{(n)}$ is constant.

Then work backwards to get that f is a polynomial of degree n

Well f' polynomial of degree k (using taylor series representation and power series differentiation) gives f is a polynomial of degree $k+1$.

Iterating gives that $f^{(n-1)}$ is linear, $f^{(n-2)}$ is quadratic ... $f^{n-(n-1)} = f'$ is polynomial of degree $n-1$ so f is polynomial of degree n.

Modification 2: Only $\text{Im}(f)$ is bounded.

We can reach the same conclusion that f is constant!

Consider $F(z) = e^{if(z)}$ which is again entire. It is also bounded because

$$|e^{-if}| = |e^{-i(\text{Re}(f)+i\text{Im}(f)})| = |e^{-i\text{Re}(f)}||e^{\text{Im}(f)}| = |e^{\text{Im}(f)}| \leq e^M$$

where M is a bound for $\text{Im}(f)$. So by Louiville's $e^{-if(z)}$ is constant, so $f(z)$ is constant too.

26 Prove the Fundamental Theorem of Algebra. (Hint: Liouville)

Sketch:

Suffices to show that a polynomial has a single root then iterate.

If $P(z)$ has no roots, then $1/P(z)$ has no poles and is entire.

Working with a polynomial, show that $1/P(z)$ is bounded.

Liouville's Thereom shows that $1/P(z)$ is constant, so $P(z)$ is constant.

Details: Let $P(z)$ be a nonconstant polynomial in $\mathbb{C}[z]$. Suppose it has no roots in $\mathbb{C}$, then $1/P(z)$ has no poles and so is entire. To show it is bounded we will split $\mathbb{C}$ into two regions. To show $1/P(z)$ bounded, meaning $|1/P(z)| \leq M$ for some M, we must show that $|P(z)| \geq 1/M$ so is bounded below. Dividing $P(z)/z^n = a_n + \frac{1}{z}(Q(1/z))$. As $z \to \infty$, the second term goes to 0, so for some R, it is less than $|a_n|/2$ and so $|P(z)/z^n| \geq |a_n|/2$ so where $|z| > R$, $|P(z)| \geq |z|^n|a_n|/2$. So in this region, $|P(z)| \geq R^n|a_n|/2$ so is bounded from below outside $|z| > R$. Inside, it is a finite region with no poles and it continuous, so will be bounded below also. (if not, there will be a sequence of points of f with $f(z_n) \to 0$ which means that $f(\lim z_n) = 0$).

since $1/P(z)$ is entire and bounded, it is constant and so $P(z)$ is constant, a contradiction. So we must have some roots.

27 Give some examples where Liouville Theorem does not apply.

entire but not bounded: $f(z) = z$ (not constant!)

bounded but not entire: piecewise functions, e.g. $f(z) = 1$ inside unit disc and $f(z) = 0$ outside (not constant!)

28 Let f be holomorphic and lie in a strip of e^z, that is $|f(z) - e^z| < R$ for some R. What can you conclude about f?

$f(z) = e^z + c$, by applying Liouvilles theorem to $g(z) = f(z) - e^z$ which is holomorphic and bounded, so it is constant and $f(z) - e^z = c$ for some $c \in \mathbb{C}$.

29 What is a holomorphic function? What properties does it have?

Definition: holomorphic = complex differentiable (derivative formula for $h \in \mathbb{C}$ $h \to 0$)

- holomorphic functions are infinitely differentiable by inductively applying cauchy integral formula
- holomorphic functions have power series expansions in little regions (local encoding of whole function)

- holomomorphic functions are determined by their behavior in small regions, giving rise to unique analytic continuation

30 What is an analytic continuation? Why is it unique?

Given f analytic on Ω an analytic continuation is an analytic (i.e. holomorphic) function on a larger region $\Omega' \supset \Omega$ that agrees with f on the smaller region

These continuations are unique because if there are two separate continuations, then they agree with each other on a small region, so they agree everywhere they are both analytic, hence they are unique.

2.5.1 Morera's Theorem

31 State and prove Morera's Theorem. What is its significance?

Morera's is a converse to Cauchy's Theorem (holomorphic $\implies \int f dz = 0$)

Morera's Theorem. *Let f be a continuous function in the open disc D. If for every triangle T contained in D,*

$$\int_T f(z)dz = 0$$

then f is holomorphic.

Significance:

A converse to Cauchy's Theorem, which gives that holomorphic $\implies \int f dz = 0$

Proof: Method is to construct, F, the anti-derivative of f which will be holomorphic, and hence infinitely differentiable. Since $F' = f$ then f is holomorphic too.

Define $F(z) = \int_{\gamma_z} f(z)dz$ where γ_z connects a fixed point to z. Then using the fact that $\int_T f dz$ vanishes, $F(z+h) - F(z) = \int_\eta f(\omega)d\omega$ for a small curve connecting z to $z+h$ and continuity of f shows that as $h \to 0$ we have $F(z+h) - F(z)/h \to f(z)$ so F is holomorphic *and* $F' = f$.

Chapter 3: Meromorphic Functions and the Logarithm

3.1 Zeros and Poles

32 What are the poles of $1/(1+z^4)$? What are their orders?

Well $1/f = 1 + z^4$ which has zeros when $z^4 = -1$, i.e. at the primitive 8th roots of unity. There are 4 distinct such roots, so each is a simple pole.

33 Find the residue of $f(x) = \frac{e^z}{z^3}$ at $z = 0$.

Method: series expansion.

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

so then

$$f(z) = \frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{3!} + \frac{z}{4!} + \cdots$$

so the residue is $1/2$ corresponding to the $1/z$ term.

3.2 The Residue Formula

34 Find the poles and residues of $1/\sin(z)$.

Poles of $1/\sin(z)$ are the zeros of $\sin(z)$. Using the formulas

$$e^{x+iy} = e^x(\cos(y) + i\sin(y)) \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

the zeros are exactly when $e^{2iz} = 1$ and which precisely when $z = n\pi$ for $n \in \mathbb{Z}$.

These are all simple zeros, can tell by differentiating or taking $\lim_{z\to n\pi} \frac{z-n\pi}{\sin(z)}$ and taking L'Hopitals to get a nonzero limit.

35 Is the residue formula usually used to compute residues when integrals are known or integrals when residues are known? Give an example.

There are other ways to compute residues (limit formula, power series, Cauchy Integral Formula) so it is often used to compute integrals, in particular complex real integrals which simplify when extended to a contour in $\mathbb{C}$.

Example: $\int_0^\infty \frac{1}{1+x^2}dx$.

Extend to the complex integral $\int_{\gamma_R} \frac{1}{1+z^2}dz$ where γ_R is the upper half plane semi circle with radius R. Let C_R denote the curved part of γ_R.

Since the residues of $1/1+z^2$ are $\pm i$, and only i lies inside our contour we can use the residue formula to compute $\int_{\gamma_R} \frac{1}{1+z^2}dz = 2\pi i \text{Res}_i(f) = 2\pi i \frac{1}{2i} = \pi$.

Now splitting up the integral we have $\int_{\gamma_R} \frac{1}{1+z^2}dz = \int_{-R}^{R} \frac{1}{1+z^2}dz + \int_{C_R} \frac{1}{1+z^2}dz$. In the limit as $R \to \infty$, this first piece becomes $\int_{\infty}^{\infty} \frac{1}{1+x^2}dx = 2\int_0^\infty \frac{1}{1+x^2}dx$. The second piece will go to zero since $\frac{1}{1+z^2}$ grows like $1/R^2$ but the length of the arc grows linearly in R, so the integral grows like $1/R$ and goes to 0. Combining this we have

$$\pi = \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{1+z^2}dz = 2\int_0^\infty \frac{1}{1+x^2}dx + 0 \implies \int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}$$

36 Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$.

Let $f(z) = \frac{1}{1+z^4}$ and γ the upper half circle of radius R.

$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \text{Re}\left(\lim_{R\to\infty}(\int_\gamma f(z)dz - \int_{C_R} f(z)dz)\right)$

By residue theorem,

$$\int_\gamma \frac{1}{1+z^4} dz = 2\pi i \operatorname{res}_{\zeta_8} f(z) = 2\pi i \frac{1}{(\zeta_8 - \zeta_8^3)(\zeta_8 - \zeta_8^5)(\zeta_8 - \zeta_8^7)} = 2\pi i \zeta_8^{-3} \frac{1}{(1-\zeta_8^2)(1-\zeta_8^4)(1-\zeta_8^6)}$$
$$= 2\pi i \zeta_8^5 \frac{1}{(1-i)(1+1)(1+i)} = 2\pi i(-1)\zeta_8 \tfrac{1}{4} = -\tfrac{1}{2}\pi i(\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2}) = \pi\frac{\sqrt{2}}{4} - i\pi\frac{\sqrt{2}}{4}$$

Note that as $|z| \to \infty$, $\frac{1}{1+z^4} \approx \frac{1}{z^4}$ so then

$$\left|\int_{C_R} f(z)dz\right| \le \frac{2\pi R}{4} ||f||_{C_R} \approx \tfrac{1}{2}\pi R \frac{1}{R^4} \to 0 \text{ as } R \to \infty$$

So then $\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \text{Re}\left(\lim_{R\to\infty}(\int_\gamma f(z)dz - \int_{C_R} f(z)dz)\right) = \text{Re}\left(\pi\frac{\sqrt{2}}{4} - i\pi\frac{\sqrt{2}}{4}\right) = \pi\frac{\sqrt{2}}{4}$

3.3 Singularities and Meromorphic Functions

37 What are the possible isolated singularities? How can we detect them?

Removable if we can define $f(z_0)$ is such a way to make f holomorphic including z_0.

Pole if $1/f(z)$ has removable singularity with $1/f(z_0) = 0$ making $1/f$ holomorphic near z_0

Essential Singularity not a pole or removable singularity

Riemann's theorem on removable singularities: If f is bounded near z_0, then z_0 is removable.

Corollary: isolated singularity is pole $\iff$ $|f(z)| \to \infty$ as $z \to z_0$.

Corollary: If f is neither bounded nor tending to ∞ then f has essential singularity.

38 What are meromorphic functions? How do they relate to rational functions?

Meromorphic functions: functions have poles at $z_0, z_1, z_2, \ldots$ points and holomorphic elsewhere with no limit points of z_i's in the region.

Holomorphic $\implies$ Meromorphic

Example on $\mathbb{C}$: $f(z) = e^z$ (holormorphic) but not meromorphic on *extended complex plane* (including the point at infinity)

Example: $f(z) = p(z)/q(z)$ rational functions

However if meromorphic on $\hat{\mathbb{C}}$ then is a rational function (ratio of polynomials), i.e. characterized (up to scaling) by its zeros/poles and their orders.

Proof Sketch:

Use behavior at ∞ to bound poles to finitely many points ($F(z) = f(1/z)$ holomorphic near 0)

Subtract off the principal parts for each pole (including ∞). This removes all poles, so entire, and no pole at infinity means bounded so this is constant. Then rewriting gives a rational funciton.

39 What is a meromorphic function? What is a pole? Let f be holomorphic in a punctured disc around 0. Is f meromorphic if we extend the domain to 0? Why or why not.

Meromorphic functions: functions have poles at $z_0, z_1, z_2, \ldots$ points and holomorphic elsewhere with no limit points of z_i's in the region.

Poles of f are places z_0 where f is defined in a punctured disc around z_0 and $1/f$ is holomophic in the entire disc when extended to 0 at z_0.

Not necessarily. It could be, in the case of $f(z) = 1/z$ which has a pole at $z = 0$ and thus is meromorphic. However it could also not have a pole or be holomorphic, for example $f(z) = e^{1/z}$ (which has an *essential singularity* at $z = 0$.) This is not defined at $z = 0$ so it is not holomorphic, and to check that it is not meromorphic we must show that it does not have a pole. If 0 were a pole, then it has order n for some n, so that $\lim_{z\to 0} z^n e^{1/z}$ exists and is finite. However looking at the expansion for e^z,

$$z^n e^{1/z} = z^n \left(1 + z^{-1} + \frac{z^{-2}}{2} + \cdots + \frac{z^{-n}}{n!} + \frac{z^{-(n+1)}}{(n+1)!} + \cdots\right) = z^n + z^{n-1} + \cdots + \frac{1}{n!} + \frac{z^{-1}}{(n+1)!} + \cdots$$

which we see does not have a finite limit as $z \to 0$ since we have a $1/z$ still. Hence this hasa no pole at 0 and is undefined at zero so is not meromorphic.

40 What is a Laurent Series? Do meromorphic functions always have one? Are they unique?

Laurent Series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$.

Existence: If f holomorphic on $r < |z-z_0| < R$ then f has a Laurent series expansion on this region. Take two circles in the annulus and form a keyhole contour containing any point z in the annulus. Cauchy integral formula $f(z)$ as the difference of the integrals along the two circles. Expand $1/w - z$ to $\frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}}$ or $\frac{1}{z-z_0}\frac{1}{1-\frac{w-z_0}{z-z_0}}$ depending on which cirlce makes the $1/1 - x$ part converge. Then massage the expansions together aligning the summation index to get $a_n = \frac{1}{2\pi i}\int_{C_n} \frac{f(w)}{(w-z_0)^{n+1}dw}$.

f **is meromorphic** then for a pole z_0, f is holomorphic on $0 < |z - z_0| < R$ where R is chosen to exclude all other poles since they have no limit points. Then f has a laurent expansion and it will have finitely many negative terms. (either by definition, or take the expansion of $1/f$ in terms of order of zero to get order of pole)

Uniqueness: If we have two expansions for $f(z)$ at z_0 then multiply by $(z - z_0)^{-k-1}$ and integrate around curve to get zero for all terms except $n = k$ where we get $2\pi i a_n$ so this shows that $a_n = b_n$ for all n.

3.4 The Argument Principles and Applications

41 What is the argument principle? Why is it called the argument principle?

f is meromorphic in Ω containing a cirlce C and it's interior with no poles/zeros along C then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \#\text{ of zeros of } f \text{ inside } C - \#\text{ of poles of } f \text{ inside } C$$

both zeros and poles counted *with multiplicity*

It is called this because if there were a logarithm of f defined in the region (not always true!) then $\log(f(z)) = \log|f(z)| + 2\pi i \arg(f(z))$ and the derivative of this would be $f'(z)/f(z)$ so the integral represents the change in argument (since the real log is well defined) around the circle.

42 What is Rouche's Theorem? Prove it.

Rouche's Theorem Let f and g be holomorphic in Ω containing a circle C and $|f(z)| > |g(z)|$ for all $z \in C$. Then f and $f + g$ have the same number of zeros inside C.

Proof. Define $f_t(z) = f(z) + tg(z)$ for $t \in [0,1]$. We want to apply the argument principle to f_t so check that it is meromorphic in the region and has no zeros/poles on C. Since f and g are both holomorphic this function is holomorphic and thus also has no poles. If it had a zero on C that $f_t(z) = f(z)+tg(z) = 0$ for some $z \in C$, so then $f(z) = tg(z)$ meaning $|f(z)| = |tg(z)| = t|g(g)| \leq |g(z)|$ since $t \leq 1$ but this contradicts our assumed inequality. Hence f_t has no zeros, so let $n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}dz$ which is the number of zeros of f_t inside C (no poles so that part drops). Since f, g are holomorphic, and f_t does not vanish on C our expression f_t'/f_t is jointly continuous in z and t so n_t will be continuous in t. However n_t is also integer valued, so that means it must be constant so $n_0 = n_1$ implying that f and $f + g$ have the same number of zeros.

43 Let Ω be a open domain with boundary C and suppose f is holomorphic mapping C to 0. What can you conclude about f?

f will have to zero as well. By the max modulus principle, the modulus of f on Ω will be bounded by the value of f on the boundary which is 0. Put another way, if f is non-constant, by the open mapping theorem $f(\Omega)$ is open and the boundary C maps to the boundary of $f(\Omega)$. However 0 is the boundary only of the set $\{0\}$ which is not open, so f must not be non-constant so $f(z) = c$ and if $c \neq 0$ then taking a limit within Ω towards the boundary gives a contradiction. Hence $f(z) = 0$ on Ω.

44 What are some consequences of the argument principle?

Argument Principle $\implies$ Rouche's Theorem $\implies$ open mapping theorem $\implies$ max modulus principle

45 What can we say topologically about holomorphic functions that are non-constant?

They are open maps (**Open Mapping Theorem**).

Proof. Let f be a holomorphic, non-constant function on Ω. Let $w_0 = f(z_0)$ be a point in the image, we want to show that there is an open neighborhood of w_0 contained in the image of f.

Since f is holomorphic and non-constant, no accumulation points of zeros, so choose $\delta > 0$ such that for all $|z - z_0| = \delta$ we have $f(z) \neq w_0$. And let $\varepsilon > 0$ be a lower bound $|f(z) - w_0| \geq \varepsilon$ for all z in the circle. Now take any w in an ε ball of w_0, that is $|w - w_0| < \varepsilon$. Then $|w - w_0| < \varepsilon \leq |f(z) - w_0|$ on the disc $|z - z_0| = \delta$ so applying rouche's theorem $f(z) - w_0$ has the same number of zeros as $g(z) = f(z) - w_0 + w - w_0 = f(z) - w$ and since there is at least one zero, we have some z' such that $0 = g(z') = f(z') - w$ so w is in the image of f for all $|w - w_0| < \varepsilon$.

46 State and prove the maximum modulus principle.

Maximum Modulus Principle. f holomorphic and non-constant on an open region Ω then f does not attain a maximum on Ω.

Proof. This is acheived by the open mapping theorem. Since f is holomorphic and non-constant in Ω, the open mapping theorem says that f is an open map.

Suppose $w_0 = f(z_0)$ is a maximum for f on Ω, then $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$. Taking a small neighborhood U of z_0, we have that $f(U)$ is an open neighborhood containing $f(z_0) = w_0$. But w_0 cannot be a maximum in an open neighborhoor and $f(U)$ lies entirely in the image of f so w_0 is not a maximum modulus for f on Ω.

47 Let f be a holomorphic from the unit disc to itself. Suppose $f(0) = 0$. What can we conclude about $f(z)/z$? Where does the image of $f(z)/z$ lie? Now suppose $|f(z_0)| = |z_0|$ for some nonzero z_0, what can you say?

First, $f(z)/z$ is also holomophic, by considering its power series expansion and noting that the constant term is 0.

By the maximum modulus principle, $f(z)/z$ attains its maximum on the boundary, where $|z| = 1$ so $|f(z)/z| \leq |f(z)/1| = |f(z)| \leq 1$ (even though f may not be defined on the boundary, we can make sense of this via limits towards the boundary). This implies that $f(z)/z : D \to D$ is also a holomorphic map from the unit disc to itself.

Now assume $|f(z_0)| = |z_0|$ for some nonzero z_0, then $|f(z_0)/z_0| = 1$ so we see that f attains its maximum modulus inside Ω which means that $f(z)/z$ must actually be constant, that is $f(z)/z = c$ and since $|c| = |f(z_0)/z_0| = 1$ we know $|c| = 1$. Rewriting we have $f(z) = cz$ where $c = e^{i\theta}$ which corresponds to a rotation of the unit disc. This is called **Schwarz's Lemma.**